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N=2 String Loops and Spectral Flow ¹

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Abstract

Based on our previous studies of the BRST cohomology of the critical $N = 2$ strings, we construct the loop measure and make explicit the role of the spectral flow at arbitrary genus and Chern class, in a holomorphic field basis. The spectral flow operator attributes to the existence of the hidden ‘small’ $N = 4$ superconformal symmetry which is non-linearly realized. We also discuss the symmetry properties of $N = 2$ string amplitudes on locally-flat backgrounds.

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1 Introduction

The $N = 2$ strings, i.e. strings with two world-sheet supersymmetries, have a long history. [1] The $N = 2$ critical strings are known to live in four dimensions, with the unphysical signature $(2, 2)$. The (smooth) target space can be e.g., $\mathbf{C}^{(1,1)}$, $\mathbf{T}^2 \otimes \mathbf{R}^2$, $\mathbf{T}^{(2,2)}$, or any Kähler and Ricci-flat Riemannian manifold with the signature $(2, 2)$. The NSR-type gauge-invariant world-sheet action is given by coupling the $N = 2$ world-sheet supergravity multiplet $(e_\alpha^a, \chi_\alpha^\pm, A_\alpha)$ to two complex $N = 2$ scalar matter multiplets $(X^\pm, \psi^\pm)^\mu$, where $\alpha = 1, 2$, and $\mu = 0, 1$. The $N = 2$ superconformal BRST gauge fixing produces (anti-commuting) conformal ghosts $(b, c)_2$, complex (commuting) $N = 2$ superconformal ghosts $(\beta^\pm, \gamma^\pm)_{3/2}$, and (anti-commuting) real $U(1)$ ghosts $(\tilde{b}, \tilde{c})_1$, where the spins appear as subscripts, and the $U(1)$ charges \pm as superscripts. The non-anomalous $N = 2$ superconformal algebra is generated by the total (with ghosts) stress tensor T_{tot} , the supercurrents G_{tot}^\pm and the $U(1)$ current J_{tot} . [2]

Contrary to the $N = 1$ string, chiral bosonization of the $U(1)$ -charged NSR fermionic fields ψ and ghosts (β, γ) depends on the field basis. In the ‘real’ basis, [3] one bosonizes real and imaginary parts of that fields, whereas it is their holomorphic and antiholomorphic combinations that are taken in the ‘holomorphic’ basis. We are going to work in the holomorphic basis, since it has the advantage of diagonalizing the local $U(1)$ symmetry. In the bosonized form, the $\psi^{\mu, \pm}$ and (β^\pm, γ^\pm) are replaced by two pairs of bosons (ϕ^\pm, φ^\pm) and two auxiliary fermion ghost systems (η^\pm, ξ^\pm) , which altogether form an extended Fock space of states containing $\mathbf{Z} \times \mathbf{Z}$ copies of the original space of fermionic states.

The natural global continuous (‘Lorentz’) symmetry of a flat $(2 + 2)$ -dimensional target space is $SO(2, 2) \cong SU(1, 1) \otimes SU(1, 1)$, but the NSR-type $N = 2$ string action has only a part of it, namely $U(1, 1) \cong U(1) \otimes SU(1, 1)$. [2] The spectrum of the critical $N = 2$ string is given by the cohomology of its BRST charge Q_{BRST} , [4] and it has only a finite number of massless physical states, all having vanishing conformal dimension and $U(1)$ charge. Further grading of the cohomology is effected by the total ghost number $u \in \mathbf{Z}$, and two picture numbers $\pi^\pm \in \frac{1}{2}\mathbf{Z}$, with $\pi^+ + \pi^- \in \mathbf{Z}$. The BRST cohomology for generic momenta consists of four classes of states for each pair (π^+, π^-) , labelled by $v \equiv u - \pi^+ - \pi^- \in \{1, 2, 2', 3\}$ and created by vertex operators V of the types cW , $\tilde{c}cW$, $c\partial cW$ and $\tilde{c}c\partial cW$, with ghost-independent W . [4]

Physical states are given by the equivalence classes of the BRST cohomology classes under the following four equivalence relations. First, cW and $c\partial cW$ are to be identified just as in the bosonic string theory. Second, \tilde{c} -type vertices get converted to others by applying the $U(1)$ ghost number-changing operator Z^0 . Third, two picture-

changing operators Z^\pm raise the picture numbers of vertices by unit amounts. And, fourth, NS- and R-type states are connected by the spectral flow [5] SFO^\pm , which move $(\pi^+, \pi^-) \rightarrow (\pi^+ \pm \frac{1}{2}, \pi^- \mp \frac{1}{2})$. Explicitly, these maps are given by [6]

$$Z^0 = \oint \tilde{b} \delta \left(\oint J_{\text{tot}} \right) , \quad Z^\pm(z) = \delta(\beta^\pm) G_{\text{tot}}^\pm , \quad SFO^\pm(z) = \exp \left(\pm \frac{1}{2} \int^z J_{\text{tot}} \right) , \quad (1)$$

they commute with Q_{BRST} but are non-trivial. In this fashion, each physical state has a representative $\tilde{c}W_{\text{can}}$ at $v = 2$, in the canonical picture $(\pi^+, \pi^-) = (-1, -1)$. Vertex operators with other ghost and picture numbers are however needed for an actual computation of string amplitudes, in order to meet certain selection rule requirements. [1] As a net result, [4] only a single scalar excitation [5] survives in an interacting theory, whereas the twisted (NS-R and R-NS) physical states, which would-be the target space ‘fermions’, decouple.³ In particular, there cannot be an interacting NSR-type critical $N = 2$ string model with a ‘space-time’ supersymmetry (*cf.* [7]).

2 $N = 2$ String Loops

To compute any n -point amplitude, one needs to sum over all genera $h \in \mathbf{Z}^+$ and $U(1)$ instanton numbers (Chern class) $c \in \mathbf{Z}$ of the Euclidean world-sheet (punctured Riemann surface) $\Sigma_{h,n}$, where

$$\chi = \frac{1}{2\pi} \int_{\Sigma} R = 2 - 2h , \quad c = \frac{1}{2\pi} \int_{\Sigma} F , \quad F = dA . \quad (2)$$

To compute the contribution for fixed h and c , one must integrate out $2h - 2 \pm c + n$ complex fermionic supermoduli of $U(1)$ charge ± 1 , respectively, and $h - 1 + n$ complex $U(1)$ moduli, to obtain an integration measure for the remaining $3h - 3 + n$ complex metric moduli.⁴ Among the complex metric moduli, $3h - 3$ ones are associated with holomorphic quadratic differentials which are dual to closed non-intersecting geodesics on Σ , and n additional ones are the positions (z_l) of vertex operators (punctures). The supermoduli count solutions to the equation $\hat{D}^\pm \chi_z^\pm \equiv (\bar{\partial} \mp iA_{\bar{z}}) \chi_z^\pm = 0$, whose number is dictated by the Riemann-Roch theorem:

$$\text{ind } \hat{D}^\pm \equiv \dim \ker \hat{D}^\pm - \dim \ker \hat{D}^{\pm\dagger} = 2(h - 1) \pm c + n . \quad (3)$$

For $h > 1$, the contributions to a positive index generically come from the first term. When the index becomes negative, $\det \hat{D}^{\pm\dagger}$ develops zero modes, which implies the

³There are also *discrete* physical states at *vanishing* momenta. They are to be accounted, if one wants to get correct factorization properties of $N = 2$ string amplitudes at vanishing momenta.

⁴The cases of the sphere and torus require some modifications, according to the index theorems.

vanishing of the corresponding correlation function. As a result, the amplitudes for $|c| > 2(h-1) + n$ all vanish. The $U(1)$ moduli space can be seen as the moduli space of flat connections on $\Sigma_{h,n}$, and it is a product of two factors. One factor is the Jacobian variety of flat $U(1)$ connections on $\Sigma_{h,0}$, dual to the homology,

$$J(\Sigma_{h,0}) = \frac{\mathbf{C}^h}{\mathbf{Z}^h + \Omega \mathbf{Z}^h} , \quad (4)$$

where Ω is the period matrix of Σ . It is parametrized by the real twists $\oint_{a_i} A$ and $\oint_{b_i} A$ around the homology cycles a_i and b_i . The other factor is the torus $\mathbf{R}^{2n-2}/\mathbf{Z}^{2n-2}$, encoding the $2n-2$ independent twists $\oint_{c_l} A$ and $\oint_{c_l} *A$ around the punctures z_l . The associated homology ‘cycles’ in the dual space are just $n-1$ independent curves connecting the punctures with a reference point z_0 .

To get the proper gauge fixing (Faddeev-Popov determinant), one should get rid of ghost zero modes by inserting anti-ghosts for each moduli direction, which come paired with Beltrami differentials, similarly to the $N=1$ string. [8] Since the total action is linear in the gauge fields and, hence, in the supermoduli and $U(1)$ moduli, one gets the additional insertions of delta-functions of the corresponding total currents. Putting all together, and using the identity $Z^\pm = \{Q_{\text{BRST}}, \xi^\pm\} = \delta(\beta^\pm) G_{\text{tot}}^\pm$, we get a product of the BRST-invariant $N=2$ picture-changing operators, as expected (*cf.* [9]),

$$\left\langle \left| \left(\oint b \right)^{3h-3+n} (Z^+)^{2h-2+c+n} (Z^-)^{2h-2-c+n} (Z^0)^{n-1} \right|^2 \prod_{i=1}^h [Z^0(a_i) Z^0(b_i)] V_1^{\text{can}} \dots V_n^{\text{can}} \right\rangle . \quad (5)$$

The picture-changing operators Z^\pm and the Z^0 can be used to convert vertex operators to other pictures and/or ghost numbers. In particular, the $U(1)$ number-changing operators Z^0 in eq. (5) enforce a projection onto charge-neutral excitations propagating across handles, which guarantees a factorization of the amplitude on neutral states only. ⁵

3 Spectral Flow

Invariance of correlation functions under the spectral flow follows from the fact that a change in monodromies for the world-sheet fermions is equivalent to a shift in the integration over $U(1)$ moduli. [5] However, this changes the loop measure by a factor

⁵The instanton-number-changing operators Z^0 have no analogue in the $N=0$ and $N=1$ strings.

$e^{-\langle \delta A, J_{\text{tot}} \rangle}$. Since δA is harmonic away from the punctures z_l , and $d * J_{\text{tot}} = 0$, we have

$$\langle \delta A, J_{\text{tot}} \rangle = \sum_{i=1}^h \left(\oint_{a_i} \delta A \oint_{b_i} * J_{\text{tot}} - \oint_{b_i} \delta A \oint_{a_i} * J_{\text{tot}} \right) + \sum_{l=1}^n \oint_{c_l} \delta A \int_{z_0}^{z_l} * J_{\text{tot}} . \quad (6)$$

which results in the twists $e^{2\pi i \lambda \int_{\gamma} * J_{\text{tot}}}$ around the cycles $\gamma = (a_i, b_i, c_l)$. The harmonicity of δA implies

$$\sum_l \text{Res}_{z_l} \delta A = 0 , \quad \text{or, equivalently,} \quad \sum_{l=1}^n \lambda_l = 0 . \quad (7)$$

In physical terms, λ_l is just the flux of the monopole field in Dirac string through z_l . On the vertices, the spectral flow is realized by

$$V_l(z_l) \longrightarrow V_l^{(\lambda_l)}(z_l) \equiv SFO(\lambda_l, z_l) \cdot V_l(z_l) , \quad (8)$$

where the ‘spectral flow operator’ takes the form [6]

$$SFO(\lambda, z) = \exp \left(2\pi i \lambda \int_{z_0}^z * J_{\text{tot}} \right) = e^{2\pi i \lambda (\phi^+ - \phi^- - \varphi^+ + \phi^- + \tilde{b}c)} . \quad (9)$$

Eq. (7) ensures that the spectral flow operator does not depend on the reference point z_0 , it is a local ⁶ and BRST-invariant operator. When $\lambda = \pm \frac{1}{2}$, it maps NS into R sector, whereas for $\lambda \in \mathbf{Z}$ it maps from the c -instanton sector to the $(c + \lambda)$ -one. Its effect on the correlation functions is

$$\langle V_1^{(\lambda_1)} \dots V_n^{(\lambda_n)} \rangle = \left\langle V_1 \dots V_n \prod_l SFO(\lambda_l) \right\rangle \stackrel{\sum \lambda_l = 0}{=} \langle V_1 \dots V_n \rangle . \quad (10)$$

equating all n -point amplitudes with the same values for h and c .

4 Amplitudes and Partition Function

Adding to the generators of the underlying $N = 2$ superconformal algebra the spectral flow operators $SFO(1, z) \equiv J^{++}(z)$ and $SFO(-1, z) \equiv J^{--}(z)$, and closing the algebra, one arrives at the ‘small’ linear $N = 4$ superconformal algebra. The presence of Z^0 insertions in the loop measure has an effect of the topological twist. After the twist, the (β^-, γ^-) ghosts acquire spin two, which is the same as that of the (b, c) system, whereas the (β^+, γ^+) ghosts get spin one, just as that of the (\tilde{b}, \tilde{c}) system. Since the respective ghosts have the opposite statistics, all the $N = 2$ ghosts *cancel*

⁶After chiral bosonization, formulations in the real and holomorphic bases are non-locally related. Therefore, the spectral flow operator is non-local in the real basis.

after the twist, at least in the partition function. [9] This paves the way for treating the $N = 2$ string measure as that of an $N = 4$ topological field theory.⁷

As far as the partition function is concerned ($n = 0$), $N = 4$ topological calculations can be put under control for the target space $\mathbf{T}^2 \otimes \mathbf{R}^2$, since the results can be recognized as topological invariants. [10] For example, the 1-loop partition function is proportional to [10] $-\log\left(\sqrt{\text{Im}\sigma\text{Im}\rho}|\eta(\sigma)|^2|\eta(\rho)|^2\right)$, which is obviously mirror-symmetric under the exchange of the Kähler and complex moduli (σ and ρ , respectively) of \mathbf{T}^2 . Similarly, the 2-loop partition function at the *maximal* instanton numbers $(2, 2)$ is σ -holomorphic, and is given by the Eisenstein series of degree 4.

As far as the scattering amplitudes are concerned, all the n -point functions beyond $n = 3$ are expected to vanish. [9] The non-trivial example is provided by the case of $n = 3$ and $h = c = 0$. The corresponding correlator is most easily computed in the $(-2, -2)$ total picture, [4]

$$\langle V(k_1, z_1)V(k_2, z_2)V(k_3, z_3) \rangle_{0,0}^{\text{left}} \propto \left(k_2^+ \cdot k_3^- - k_2^- \cdot k_3^+\right), \quad (11a)$$

and it is $U(1, 1)$ invariant. [5] The case of $n = 3$, $h = 0$ and $c = 1$ is a bit more complicated,

$$\left\langle \tilde{c}cW_{(-1,-1)}(k_1, z_1)cW_{(0,-1)}(k_2, z_2)cW_{(0,-1)}(k_3, z_3) \right\rangle_{0,1}^{\text{left}} \propto \varepsilon^{\mu\nu}k_2^{\mu-}k_3^{\nu-}, \quad (11b)$$

and it is only $SO(1, 1)$ invariant. Our final example is the 1-loop 3-point function at $c = 0$, which reads (*cf.* [11])

$$\left\langle \int W_{(0,0)} \int W_{(0,0)} \int W_{(0,0)} \right\rangle_{(1,0)} \propto \left(k_2^{[+} \cdot k_3^{-]}\right)^3, \quad (12)$$

and it is non-vanishing, and local in momenta.

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⁷ There are indications [10] that the latter is the *holomorphic* Yang-Mills theory in the limit $N_{\mathbf{c}} \rightarrow \infty$.

References

- [1] S. V. Ketov, *Conformal Field Theory*, Singapore: World Scientific, 1995.
- [2] S. V. Ketov, O. Lechtenfeld and A. J. Parkes, *Phys. Rev.* **D51** (1995) 2872.
- [3] C. Pope, *The $N=2$ string in a real spacetime structure*, Talk at the International Workshop on Strings, Gravity and Related Topics, June 29–30, 1995, ICTP, Trieste, Italy; to appear in the Proceedings (World Scientific).
- [4] J. Bischoff, S. V. Ketov and O. Lechtenfeld, *Nucl. Phys.* **B438** (1995) 373.
- [5] H. Ooguri and C. Vafa, *Nucl. Phys.* **B361** (1991) 469.
- [6] S. V. Ketov and O. Lechtenfeld, *Phys. Lett.* **353B** (1995) 463.
- [7] W. Siegel, *Phys. Rev. Lett.* **69** (1992) 1493; *Phys. Rev.* **D47** (1993) 2504.
- [8] E. Verlinde and H. Verlinde, *Nucl. Phys.* **B288** (1987) 357.
- [9] N. Berkovits and C. Vafa, *Nucl. Phys.* **B433** (1995) 123.
- [10] H. Ooguri and C. Vafa, *All Loop $N = 2$ String Amplitudes*, Harvard preprint HUTP-95/A017; hep-th/9505183.
- [11] M. Bonini, E. Gava and R. Iengo, *Mod. Phys. Lett.* **A6** (1991) 795.